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Reply to Attn of : EG-96-02

TO: Distribution

FROM: EG2/Carlos Roithmayr

SUBJECT: Contributions of Spherical Harmonics to Magnetic and Gravitational Fields

The enclosed document presents expressions for gravitational force used in FORTRAN subroutines that will form a new gravity model for the Space Station Multi Rigid Body Simulation. Gravitational forces calculated with these routines and an ADA program generously provided by MDA/R. Gottlieb have been compared for harmonics up to degree and order 4, and are identical to 14 decimal places. Corresponding MATLAB routines that produce results also identical to 14 decimal places have been written, and are available from the author.

The routines are written so that terms common to gravitational and magnetic fields are computed once, and are then available to use in both models.

Original signed by

Carlos Roithmayr

Enclosure

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Contributions of Spherical Harmonics to Magnetic and Gravitational Fields

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Introduction

Gravitational forces are of cardinal importance in the dynamics of spacecraft; magnetic attractions sometime play a significant role also, as was the case with the Long Duration Exposure Facility, and as is now true for the first segment of Space Station Freedom. Both satellites depend on gravitational moment and a device known as a magnetic damper to stabilize their orientation.

Magnetic fields are mathematically similar to gravitational fields in one important respect: each can be regarded as a gradient of a potential function that, in turn, can be described as an infinite series of spherical harmonics. Consequently, the two fields can be computed, in part, with quantities that need only be evaluated once, resulting in a savings of time when both fields are needed.

The objective of this material is to present magnetic field and gravitational force expressions, and point out the terms that belong to both—this is accomplished in Sections 1 and 2. Section 3 contains the deductive reasoning with which one obtains the expressions of interest. Finally, examples in Section 4 show these equations can be used to reproduce others that arise in connection with special cases such as the magnetic field produced by a tilted dipole, and gravitational force exerted by an oblate spheroid. The mathematics are discussed in the context of terrestrial fields; however, by substituting appropriate constants, the results can be made applicable to fields belonging to other celestial bodies.

The expressions presented here share the characteristics of algorithms set forth in Refs. [1]–[4] for computing gravitational force. In particular, computation is performed speedily by means of recursion formulae, and the expressions do not suffer from the shortcoming of a singularity when evaluated at points that lie on the polar axis.

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1 Contributions to Magnetic Field

The magnetic field vector $\mathbf{B}$ at a point Q above the Earth's surface can be expressed as the sum

$$\mathbf{B} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \mathbf{B}_{n,m} \tag{1}$$

where $\mathbf{B}_{n,m}$ represents the contribution to $\mathbf{B}$ of the spherical harmonic of degree n and order m, and is given by

$$\begin{aligned}\mathbf{B}_{n,m} = \frac{K_{n,m}\, a^{n+2}}{R^{n+m+1}} \Bigg\{ & \frac{g_{n,m}\mathcal{C}_m + h_{n,m}\mathcal{S}_m}{R} \left[(\mathrm{s}_\lambda A_{n,m+1} + (n+m+1)A_{n,m})\, \hat{\mathbf{r}} - A_{n,m+1}\hat{\mathbf{e}}_3 \right] \\ & - mA_{n,m} \left[(g_{n,m}\mathcal{C}_{m-1} + h_{n,m}\mathcal{S}_{m-1})\hat{\mathbf{e}}_1 + (h_{n,m}\mathcal{C}_{m-1} - g_{n,m}\mathcal{S}_{m-1})\hat{\mathbf{e}}_2 \right] \Bigg\}\end{aligned} \tag{2}$$

Here, a is the mean radius of the Earth (6371 km: p. 25, Ref. [5]) adopted for the International Geomagnetic Reference Field; R is the magnitude of $\mathbf{R}$, the position vector from the geometric center of the Earth to Q; $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{R}$; and $g_{n,m}$ and $h_{n,m}$ represent Gauss coefficients of degree n and order m. Mutually perpendicular unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ are fixed in the Earth: $\hat{\mathbf{e}}_1$ lies in the equatorial plane parallel to a line intersecting Earth's geometric center and the Greenwich meridian, $\hat{\mathbf{e}}_3$ is in the direction of the north polar axis, and $\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1$.

The coefficients $K_{n,m}$ in Eq. (2) are given by

$$K_{n,0} = 1 \qquad (n = 1, \ldots, \infty) \tag{3}$$

whenever $m = 0$ (in addition, $K_{1,1} = 1$); otherwise, they can be obtained with one of two recursion formulae:

$$K_{n,m} = \left(\frac{n-m}{n+m} \right)^{\frac{1}{2}} K_{n-1,m} \qquad [m = 1, \ldots, \infty;\ n \geq (m+1)] \tag{4}$$

$$K_{n,m} = [(n+m)(n-m+1)]^{-\frac{1}{2}} K_{n,m-1} \qquad (m = 2, \ldots, \infty;\ n \geq m) \tag{5}$$

$A_{n,m}$ and $A_{n,m+1}$ are referred to as derived Legendre polynomials of degree n, and orders m and $m+1$ respectively. The notation $A_{n,m}(u)$ is used to indicate that $A_{n,m}$ is, in general, a function of an argument u; however, for the sake of clarity, the arguments of $A_{n,m}$ and $A_{n,m+1}$ do not appear explicitly in Eq. (2). When $n = m$, the derived Legendre polynomial is given by a simple expression that can be written in recursive form.

$$\begin{aligned} A_{n,n} &= (1)(3)(5)\cdots(2n-1) \qquad (n = 1, \ldots, \infty) \\ &= (2n-1)A_{n-1,n-1} \qquad (n = 2, \ldots, \infty) \end{aligned} \tag{6}$$

It should be noted that $A_{1,1}$, $A_{2,2}, \ldots$ are not functions of an argument; therefore, they need only be computed once to provide the seeds for a recursion relation shown in Ref. [6]

to give numerically stable and precise results:

$$A_{n,m}(u) = \frac{1}{n-m}\left[(2n-1)uA_{n-1,m}(u) - (n+m-1)A_{n-2,m}(u)\right]$$
$$[m = 0, \ldots, \infty;\ n \geq (m+1)] \qquad (7)$$

$A_{n,m}(u)$ vanishes when $m > n$, and $A_{0,0}(u) = 1$. Arguments of the derived Legendre polynomials in Eq. (2) are $\sin\lambda$ (shortened to s_λ), where λ is the geographic latitude of Q. $\mathrm{s}_\lambda = \hat{\mathbf{r}} \cdot \hat{\mathbf{e}}_3$.

The symbols $\mathcal{S}_m$ and $\mathcal{C}_m$ appearing in Eq. (2) will be defined presently. Their values for $m = 0$ and $m = 1$,

$$\mathcal{S}_0 = 0, \qquad \mathcal{C}_0 = 1 \qquad (8)$$

$$\mathcal{S}_1 = \mathbf{R} \cdot \hat{\mathbf{e}}_2, \qquad \mathcal{C}_1 = \mathbf{R} \cdot \hat{\mathbf{e}}_1 \qquad (9)$$

provide a means of starting the recursion relations given on p. 4 of Ref. [2].

$$\mathcal{S}_m = \mathcal{S}_1\mathcal{C}_{m-1} + \mathcal{C}_1\mathcal{S}_{m-1}, \qquad \mathcal{C}_m = \mathcal{C}_1\mathcal{C}_{m-1} - \mathcal{S}_1\mathcal{S}_{m-1} \qquad (10)$$

2 Contributions to Gravitational Field

Like the magnetic field vector, the gravitational force per unit mass $\mathbf{f}$ exerted by the Earth at a point Q above the Earth's surface, can be regarded as a sum:

$$\mathbf{f} = -\frac{\mu}{R^2}\hat{\mathbf{r}} + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\mathbf{f}_{n,m} \qquad (11)$$

where μ, the gravitational parameter, is the product of G, the universal gravitational constant, and M, the Earth's mass; the position vector from Earth's mass center to Q has magnitude R and the direction of unit vector $\hat{\mathbf{r}}$; and $\mathbf{f}_{n,m}$ represents the contribution to $\mathbf{f}$ of the spherical harmonic of degree n and order m, given by

$$\mathbf{f}_{n,m} = \frac{\mu R_\oplus{}^n}{R^{n+m+1}}\left\{\frac{C_{n,m}\mathcal{C}_m + S_{n,m}\mathcal{S}_m}{R}\left[A_{n,m+1}\hat{\mathbf{e}}_3 - \left(\mathrm{s}_\lambda A_{n,m+1} + (n+m+1)A_{n,m}\right)\hat{\mathbf{r}}\right]\right.$$
$$\left. + mA_{n,m}\left[(C_{n,m}\mathcal{C}_{m-1} + S_{n,m}\mathcal{S}_{m-1})\hat{\mathbf{e}}_1 + (S_{n,m}\mathcal{C}_{m-1} - C_{n,m}\mathcal{S}_{m-1})\hat{\mathbf{e}}_2\right]\right\} \qquad (12)$$

$R_\oplus$ is the mean equatorial radius of the Earth (6,378,139 m), and $C_{n,m}$ and $S_{n,m}$ are gravitational coefficients of degree n and order m. Mutually perpendicular unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ have the same directions as in Eq. (2). $A_{n,m}$, $\mathcal{S}_m$, and $\mathcal{C}_m$ have the same meanings as in Eq. (2); consequently, they need only be computed once when constructing magnetic and gravitational fields.

It is not too difficult to show that Eq. (11) and Eq. (12), which has the form $\mathbf{f}_{n,m} = x_{n,m}\hat{\mathbf{e}}_1 + y_{n,m}\hat{\mathbf{e}}_2 + z_{n,m}\hat{\mathbf{e}}_3 + w_{n,m}\hat{\mathbf{r}}$ [Eq. (2) has this form as well], are in concert with Eq. (31) of Ref. [1], whose form is $\mathbf{f} = X\hat{\mathbf{e}}_1 + Y\hat{\mathbf{e}}_2 + Z\hat{\mathbf{e}}_3 + W\hat{\mathbf{r}}$: in other words, $W = -\mu/R^2 + \sum_{n=2}^{\infty}\sum_{m=0}^{n} w_{n,m}$, $X = \sum_{n=2}^{\infty}\sum_{m=0}^{n} x_{n,m}$, and so forth. (The sum to the right of the second $=$ sign in Eqs. (30a) of Ref. [1] should be preceded by a minus sign.)

3 Derivations

Eq. (2) is similar in form to Eq. (12) because $\mathbf{B}$ and $\mathbf{f}$ are each obtained by differentiating a potential function that is expressed in terms of spherical harmonics. A potential function associated with the magnetic field is given below, followed by steps in the differentiation process that results in Eq. (2). Next, a gravitational potential function is displayed. The differentiation process that leads to Eq. (12) is virtually identical to that used with the magnetic potential; therefore, it does not bear repeating.

3.1 Magnetic Field

The magnetic field vector $\mathbf{B}$ is related to V_β, a scalar geomagnetic potential, by the equation

$$\mathbf{B} = -\nabla V_\beta \tag{13}$$

where ∇ denotes differentiation with respect to $\mathbf{R}$: $\mathbf{B}$ is the negative of the spatial gradient of V_β. Section 2.9 of Ref. [7] contains a thorough explanation of how one differentiates with respect to a vector. The potential V_β satisfies Laplace's equation and, consequently, is expressed in Eq. (2.13) of Ref. [5], and in Eq. (9.0,6) of Ref. [8], as an infinite series of spherical harmonics.

$$V_\beta = a \sum_{n=1}^{\infty} \left(\frac{a}{R}\right)^{n+1} \sum_{m=0}^{n} P_n^m(\cos\theta)\Big[g_{n,m}\cos m\phi + h_{n,m}\sin m\phi\Big] \tag{14}$$

where a, R, $g_{n,m}$, and $h_{n,m}$ have the same meanings as in Eq. (2), and ϕ is the geographic longitude of Q measured Eastward from the Greenwich meridian. $P_n^m(\cos\theta)$ are Schmidt functions (they will be discussed directly) whose arguments are $\cos\theta$, where θ is the geographic co-latitude of Q. Now, Eq. (14) can be rewritten as

$$V_\beta = \sum_{n=1}^{\infty} \sum_{m=0}^{n} V_{n,m} \tag{15}$$

so long as $V_{n,m}$ is defined as

$$V_{n,m} \triangleq a \left(\frac{a}{R}\right)^{n+1} P_n^m(\cos\theta)\Big[g_{n,m}\cos m\phi + h_{n,m}\sin m\phi\Big] \tag{16}$$

For the sake of clarity, we dispense with the subscript β for $V_{n,m}$. Thus, in view of Eqs. (13) and (15), $\mathbf{B}$ can be expressed as in Eq. (1)

$$\mathbf{B} = -\nabla V_\beta = -\nabla \sum_{n=1}^{\infty} \sum_{m=0}^{n} V_{n,m} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \mathbf{B}_{n,m} \tag{17}$$

with the contribution to $\mathbf{B}$ of the spherical harmonic of degree n and order m defined simply as

$$\mathbf{B}_{n,m} \triangleq -\nabla V_{n,m} \tag{18}$$

The Schmidt functions P_n^m are of degree n and order m, and are related [Ref. [9], Chapter 9, Eq. (13)] to $P_{n,m}$, associated Legendre functions of the first kind.

$$P_n^m(\cos\theta) = \begin{cases} \left[2\frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} P_{n,m}(\cos\theta) & (m = 1, \ldots, n) \\ P_{n,m}(\cos\theta) & (m = 0) \end{cases} \tag{19}$$

Angle θ, the co-latitude of Q, is the complement of λ, the latitude of Q; therefore, the argument $\cos\theta$ of P_n^m and $P_{n,m}$ can be replaced with s_λ, which is shorthand for $\sin\lambda$. Products of $P_{n,m}(\mathrm{s}_\lambda)$ with $\cos m\phi$, or with $\sin m\phi$, are referred to as tesseral harmonics of degree n and order m: they are functions of latitude and longitude. Tesseral harmonics of order zero are called zonal harmonics, and are independent of longitude.

In the interest of economy in written symbols, the coefficients of $P_{n,m}$ in Eqs. (19) can be labeled $K_{n,m}$ [see Eq. (2)] and defined as

$$K_{n,m} \triangleq \begin{cases} \left[2\frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} & (m = 1, \ldots, n) \\ 1 & (m = 0) \end{cases} \tag{20}$$

The recursive relationships in Eqs. (4) and (5) follow from the first of these definitions; Eqs. (3) from the second.

The associated Legendre functions $P_{n,m}$ are, in turn, related to derived Legendre polynomials $A_{n,m}$ by Eq. (9b) of Ref. [6] (see also p. 3 of Ref. [2]).

$$P_{n,m}(\mathrm{s}_\lambda) = (\cos\lambda)^m A_{n,m}(\mathrm{s}_\lambda) = \left(\frac{\rho}{R}\right)^m A_{n,m}(\mathrm{s}_\lambda) \tag{21}$$

where ρ is related to $\mathbf{R}$ by the definition

$$\rho^2 \triangleq (\mathbf{R} \cdot \hat{\mathbf{e}}_1)^2 + (\mathbf{R} \cdot \hat{\mathbf{e}}_2)^2 \tag{22}$$

In words, ρ is the magnitude of the projection of $\mathbf{R}$ onto Earth's equatorial plane.

The symbols $\mathcal{S}_m$ and $\mathcal{C}_m$ appearing in Eq. (2) are defined on p. 4 of Ref. [2].

$$\mathcal{S}_m \triangleq \rho^m \sin m\phi, \qquad \mathcal{C}_m \triangleq \rho^m \cos m\phi \tag{23}$$

where ϕ has the same meaning as in Eq. (14).

Differentiation of $V_{n,m}$ with respect to $\mathbf{R}$ is facilitated by substitution from Eqs. (19), (20), (21), and (23) into (16).

$$\begin{aligned} V_{n,m} &= a\left(\frac{a}{R}\right)^{n+1} K_{n,m}\left(\frac{\rho}{R}\right)^m A_{n,m}(\mathrm{s}_\lambda)\left[g_{n,m}\cos m\phi + h_{n,m}\sin m\phi\right] \\ &= K_{n,m}\, a^{n+2} A_{n,m}\frac{g_{n,m}\mathcal{C}_m + h_{n,m}\mathcal{S}_m}{R^{n+m+1}} \end{aligned} \tag{24}$$

$K_{n,m}$, a, $g_{n,m}$, and $h_{n,m}$ are independent of $\mathbf{R}$, so obtaining an expression for the gradient $\nabla V_{n,m}$ becomes a matter of producing the four gradients $\nabla R^{-(n+m+1)}$, $\nabla A_{n,m}$, $\nabla \mathcal{C}_m$, and $\nabla \mathcal{S}_m$.

Eqs. (2.9.7)–(2.9.9) of Ref. [7] provide the machinery with which all the required gradients can be formed:

$$\nabla \mathbf{R} = \underline{\mathrm{U}} \tag{25}$$

$$\nabla R = \hat{\mathbf{r}} \tag{26}$$

$$\nabla \hat{\mathbf{r}} = R^{-1}(\underline{\mathrm{U}} - \hat{\mathbf{r}}\hat{\mathbf{r}}) \tag{27}$$

where $\underline{\mathrm{U}}$ is the unit dyadic. (When $\underline{\mathrm{U}}$ is pre or post multiplied by any vector $\mathbf{v}$, the result is $\mathbf{v}$, that is, $\underline{\mathrm{U}} \cdot \mathbf{v} = \mathbf{v} \cdot \underline{\mathrm{U}} = \mathbf{v}$.) The first gradient is fashioned easily with the help of Eq. (26).

$$\nabla R^{-(n+m+1)} = -(n+m+1)R^{-(n+m+2)}\hat{\mathbf{r}} \tag{28}$$

Referring to Eq. (10) of Ref. [6], one can write $\nabla A_{n,m}(\mathrm{s}_\lambda) = A_{n,m+1}\nabla \mathrm{s}_\lambda$, express s_λ in terms of $\hat{\mathbf{r}}$ and $\hat{\mathbf{e}}_3$ ($\hat{\mathbf{e}}_3$ is independent of $\mathbf{R}$) in order to expedite differentiation with respect to $\mathbf{R}$ by means of Eq. (27),

$$\nabla \mathrm{s}_\lambda = \nabla(\hat{\mathbf{r}} \cdot \hat{\mathbf{e}}_3) = R^{-1}(\underline{\mathrm{U}} - \hat{\mathbf{r}}\hat{\mathbf{r}}) \cdot \hat{\mathbf{e}}_3 = [\hat{\mathbf{e}}_3 - (\hat{\mathbf{r}} \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{r}}]/R = (\hat{\mathbf{e}}_3 - \mathrm{s}_\lambda \hat{\mathbf{r}})/R \tag{29}$$

and obtain the second gradient.

$$\nabla A_{n,m}(\mathrm{s}_\lambda) = (\hat{\mathbf{e}}_3 - \mathrm{s}_\lambda \hat{\mathbf{r}})A_{n,m+1}/R \tag{30}$$

Finally, the remaining two gradients can be produced from Eqs. (23) with the aid of Eqs. (22) and (25),

$$\nabla \mathcal{S}_m = m(\mathcal{S}_{m-1}\hat{\mathbf{e}}_1 + \mathcal{C}_{m-1}\hat{\mathbf{e}}_2), \qquad \nabla \mathcal{C}_m = m(\mathcal{C}_{m-1}\hat{\mathbf{e}}_1 - \mathcal{S}_{m-1}\hat{\mathbf{e}}_2) \tag{31}$$

as is suggested by Eqs. (7) in Ref. [2].

The foregoing material puts one into position to express $\mathbf{B}_{n,m}$ as in Eq. (2) by substituting from Eq. (24) into (18), and then employing the right hand members of Eqs. (28), (30), and (31).

3.2 Gravitational Field

The force $\mathbf{f}$ in a gravitational field is analagous to the vector $\mathbf{B}$ in a magnetic field: both are spatial gradients of scalar potential functions. At a point Q above Earth's surface, a particle of unit mass is subject to a gravitational force $\mathbf{f}$ given by

$$\mathbf{f} = \nabla V_\gamma \tag{32}$$

where V_γ is a scalar gravitational potential, and ∇ denotes differentiation with respect to $\mathbf{R}$, the position vector from Earth's mass center to Q. Like V_β, V_γ satisfies Laplace's equation and can be expressed as an infinite sum of spherical harmonics. Eq. (2.13.12) of

Ref. [7] is one such example, written in the form adopted as a standard by the International Astronomical Union:

$$
\begin{aligned}
V_\gamma &= \frac{\mu}{R}\left\{1+\sum_{n=2}^{\infty}\left(\frac{R_\oplus}{R}\right)^n\sum_{m=0}^{n}P_{n,m}(\sin\lambda)\Big[C_{n,m}\cos m\phi+S_{n,m}\sin m\phi\Big]\right\} \\
&= \frac{\mu}{R}+\sum_{n=2}^{\infty}\sum_{m=0}^{n}V_{n,m}
\end{aligned} \tag{33}
$$

where μ, $R_\oplus$, $C_{n,m}$, and $S_{n,m}$ have the same meanings as in Eq. (12), and Eqs. (21) and (23) can be engaged to define $V_{n,m}$:

$$
V_{n,m} \triangleq \mu {R_\oplus}^n A_{n,m}\frac{C_{n,m}\mathcal{C}_m+S_{n,m}\mathcal{S}_m}{R^{n+m+1}} \tag{34}
$$

Thus, Eq. (11) is the result of substituting from Eq. (33) into (32).

$$
\mathbf{f} = \nabla\frac{\mu}{R}+\sum_{n=2}^{\infty}\sum_{m=0}^{n}\nabla V_{n,m} = -\frac{\mu}{R^2}\hat{\mathbf{r}}+\sum_{n=2}^{\infty}\sum_{m=0}^{n}\mathbf{f}_{n,m} \tag{35}
$$

and the expression for $\mathbf{f}_{n,m}$ in Eq. (12) follows from differentiation of the right hand member of Eq. (34) [which is of the same form as Eq. (24)] in the much the same way as $\mathbf{B}_{n,m}$ results from differentiation of the right side of Eq. (24). μ, $R_\oplus$, $C_{n,m}$, and $S_{n,m}$ are all independent of $\mathbf{R}$.

4 Examples

Although Eqs. (2) and (12) are intended to be evaluated numerically, rather than analytically, they can be used to produce recognizable expressions in a few important special cases. In this Section, Eqs. (2) and (12) are exercised through applications to a magnetic field associated with a tilted dipole, to the gravitational force exerted by a planet whose mass is distributed symmetrically about its polar axis, and to the gravitational attraction of an oblate spheroid—all of which have been dealt with elsewhere.

4.1 Magnetic Field of a Tilted Dipole

The geomagnetic field can be modeled as a tilted dipole by adding the contributions of spherical harmonics of degree 1, and orders 0 and 1. According to Ref. [5], approximately 90% of the magnetic field at the Earth's surface is described by this model. The derived Legendre polynomials identified with the dipole are [see Eqs. (6) and (7)] simply

$$
A_{1,0}(\mathrm{s}_\lambda) = \mathrm{s}_\lambda, \qquad A_{1,1}(\mathrm{s}_\lambda) = 1, \qquad A_{1,2}(\mathrm{s}_\lambda) = 0 \tag{36}
$$

and Eq. (20) yields the coefficients

$$
K_{1,0} = K_{1,1} = 1 \tag{37}
$$

Substitution from Eqs. (36), (37), (8), and (9) into (2) leads, for $n = 1$, $m = 0$, and $m = 1$, to

$$\mathbf{B}_{1,0} = \left(\frac{a}{R}\right)^3 g_{1,0}(3\mathrm{s}_\lambda \hat{\mathbf{r}} - \hat{\mathbf{e}}_3) \tag{38}$$

$$\mathbf{B}_{1,1} = \left(\frac{a}{R}\right)^3 [3(g_{1,1}\hat{\mathbf{r}} \cdot \hat{\mathbf{e}}_1 + h_{1,1}\hat{\mathbf{r}} \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{r}} - (g_{1,1}\hat{\mathbf{e}}_1 + h_{1,1}\hat{\mathbf{e}}_2)] \tag{39}$$

and then, after recognizing that $\mathrm{s}_\lambda = \hat{\mathbf{r}} \cdot \hat{\mathbf{e}}_3$, to

$$\mathbf{B}_{1,0} + \mathbf{B}_{1,1} = \left(\frac{a}{R}\right)^3 [3(\hat{\mathbf{r}} \cdot \mathbf{M})\hat{\mathbf{r}} - \mathbf{M}] \tag{40}$$

which is in agreement with Eq. (H-22) of Ref. [10] when $\mathbf{M}$, the terrestrial dipole moment, is defined as $\mathbf{M} \triangleq g_{1,1}\hat{\mathbf{e}}_1 + h_{1,1}\hat{\mathbf{e}}_2 + g_{1,0}\hat{\mathbf{e}}_3$.

4.2 Gravitational Force Exerted by an Axisymmetric Body

An expression for the gravitational force per unit mass, exerted by an axisymmetric body, is given in Ref. [11], and can be reproduced from Eqs. (11) and (12). The tesseral harmonics vanish from the gravitational potential as a consequence of axial symmetry, leaving only the zonal harmonics. Working only with terms of order 0, and substituting from Eq. (8) into (12), we get

$$\mathbf{f} = -\frac{\mu}{R^2}\hat{\mathbf{r}} + \sum_{n=2}^{\infty} \mathbf{f}_{n,0} = -\frac{\mu}{R^2}\hat{\mathbf{r}} - \mu \sum_{n=2}^{\infty} J_n \frac{{R_\oplus}^n}{R^{n+2}} \left\{A_{n,1}\hat{\mathbf{e}}_3 - [\mathrm{s}_\lambda A_{n,1} + (n+1)A_{n,0}]\,\hat{\mathbf{r}}\right\} \tag{41}$$

where the zonal harmonic coefficients, $C_{n,0}$, have been renamed: $C_{n,0} \triangleq -J_n$. The derived Legendre polynomials are defined in Eq. (14) of Ref. [1] as $A_{n,m}(u) \triangleq d^m P_n(u)/du^m$, where $P_n(u)$ is a Legendre polynomial of degree n, with argument u. Denoting the first derivative (with respect to u) of $P_n(u)$ as $P'_n(u)$ leads to

$$\mathbf{f} = -\frac{\mu}{R^2}\hat{\mathbf{r}} - \mu \sum_{n=2}^{\infty} J_n \frac{{R_\oplus}^n}{R^{n+2}} \left\{P'_n\hat{\mathbf{e}}_3 - \left[\mathrm{s}_\lambda P'_n + (n+1)P_n\right]\hat{\mathbf{r}}\right\} \tag{42}$$

which, upon use of the recursion relation $P'_{n+1}(u) = uP'_n(u) + (n+1)P_n(u)$, becomes the expression given on p. 407 of Ref. [11]:

$$\mathbf{f} = -\frac{\mu}{R^2}\left[\hat{\mathbf{r}} + \sum_{n=2}^{\infty} J_n \left(\frac{R_\oplus}{R}\right)^n (P'_n\hat{\mathbf{e}}_3 - P'_{n+1}\hat{\mathbf{r}})\right] \tag{43}$$

4.3 Gravitational Force Exerted by an Oblate Spheroid

An oblate spheroid is an example of an axisymmetric body, and is often used to model the Earth. A celestial body's oblateness is represented by a zonal harmonic of degree 2, and is responsible for precessions in a satellite's orbit plane and argument of perigee. The derived Legendre polynomials identified with oblateness are [see Eqs. (6) and (7)] just

$$A_{2,0}(\mathrm{s}_\lambda) = \frac{1}{2}(3{\mathrm{s}_\lambda}^2 - 1), \qquad A_{2,1}(\mathrm{s}_\lambda) = 3\mathrm{s}_\lambda \tag{44}$$

Substitution from Eqs. (44) and (8) into Eq. (12) yields, for $n = 2$ and $m = 0$,

$$\mathbf{f}_{2,0} = -\mu J_2 \frac{{R_\oplus}^2}{R^4}\left[3\mathrm{s}_\lambda \hat{\mathbf{e}}_3 + \frac{3}{2}(1 - 5{\mathrm{s}_\lambda}^2)\hat{\mathbf{r}}\right] \tag{45}$$

This expression for the contribution of oblateness to the gravitational force per unit mass, exerted by Earth, is in agreement with the vector equation (8.28) in Ref. [12] and, after a little manipulation, with the scalar equations for a_x, a_y, and a_z given on p. 3-1 of Ref. [3]. [$a_x = (-\mu\hat{\mathbf{r}}/R^2 + \mathbf{f}_{2,0}) \cdot \hat{\mathbf{e}}_1$, $a_y = (-\mu\hat{\mathbf{r}}/R^2 + \mathbf{f}_{2,0}) \cdot \hat{\mathbf{e}}_2$, and $a_z = (-\mu\hat{\mathbf{r}}/R^2 + \mathbf{f}_{2,0}) \cdot \hat{\mathbf{e}}_3$.]

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